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ON THE DIFFERENTIAL EQUATION

$$\frac{dy}{dx} + y^2 + Py + Q = 0.$$

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1. The peculiarity of the equation here considered, in which P and Q are functions of x, is its reducibility to a linear form of the second order, namely,

$$\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Q = 0,$$

by the transformation $y = \frac{1}{w}\frac{dw}{dx}$. The equation of the first order includes Riccati's equation, which may, without loss of generality, be written

$$\frac{dy}{dx} + y^2 = cx^m;$$

and the reduction to the linear form has been extensively used in the treatment of that equation, for example, in Cayley's solution.* The present paper presents the relation between the equations in their general forms from quite a different point of view.

2. Starting with the linear equation of the second order

$$\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Q = 0, \qquad (1)$$

and denoting $\frac{d}{dx}$ by D, let the equation be written in the symbolic form

$$(D^2 + P.D + Q)\,w = 0; \qquad (2)$$

and let the attempt be made to solve the equation, by resolving the operator into symbolic factors of the first order: for if we can reduce the equation to the form

$$[D - \varphi(x)]\,[D - f(x)]\,w = 0, \qquad (3)$$

we can, by the successive solution of two linear equations of the first order, express w as a function of x. Denote the function $f(x)$ by y. It is readily seen that $-\varphi(x)$ must then be $P + y$, so that the equation is

$$(D + P + y)(D - y)\,w = 0, \qquad (4)$$

in which y denotes an unknown function of x. This equation is to be identical with equation (2); expanding, it becomes

$$D^2w + P.Dw - (Dy + Py + y^2)\,w = 0,$$

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* *Phil. Mag.*, Fourth Series, Vol. 36, pp. 348–351.

and comparing this with equation (2) we have for the determination of y

$$\frac{dy}{dx}+y^2+Py+Q=0, \tag{5}$$

an equation of the first order, but not linear.

3. Suppose now that y is any particular integral of equation (5); then if w satisfies the equation

$$(D-y)\,w=0, \tag{6}$$

it will plainly satisfy equation (4), that is to say, equation (1). Solving equation (6), we find

$$w=ce^{\int y\,dx}. \tag{7}$$

Thus to every particular integral of equation (5) there corresponds an integral of equation (1), and if we have two particular integrals of (5), y_1 and y_2, we have two *independent* integrals of (1), and therefore its complete integral

$$w=c_1e^{\int y_1dx}+c_2e^{\int y_2dx},$$

without the necessity of solving successively two linear equations as indicated in equation (4). But, if we possess only one particular integral y_1 of (5), then w_1 is defined by (7), and by putting in (4)

$$(D-y_1)\,w=v, \tag{8}$$

we have

$$(D+P+y_1)\,v=0;$$

whence

$$v=c_2\,e^{-\int Pdx}\,e^{-\int y_1dx}=\frac{c_2}{w_1}e^{-\int Pdx};$$

and finally, by the solution of the linear equation (8), we obtain the complete integral

$$w=c_1w_1+c_2w_1\int e^{-\int Pdx}\frac{dx}{w_1^{\,2}},$$

in which the second integral is expressed in its usual form in terms of the integral w_1, supposed known.

4. Equation (6) may be written

$$y=\frac{1}{w}\frac{dw}{dx}=\frac{d}{dx}\log w, \tag{6}$$

thus expressing the integral of equation (5) in terms of that of the linear equation (1). Because, therefore, the integral of the latter equation is of the form

$$w=c_1X_1+c_2X_2,$$

that of the former is of the form

$$y = \frac{c_1X_1' + c_2X_2'}{c_1X_1 + c_2X_2},$$

in which X_1 and X_2 are functions of x, and X_1', X_2' their derivatives; or, putting c in place of the ratio $c_2 : c_1$,

$$y = \frac{X_1' + cX_2'}{X_1 + cX_2}, \tag{10}$$

which contains, as it should, but one arbitrary constant. The mode in which the constant c enters the complete integral is the same as in the special case of Riccati's equation, and is the characteristic of the form of differential equation under consideration.

5. The equation (5) is no easier of solution in any given case than the linear equation (1) of the second order; so that in general, as well as in the special case of Riccati's equation, it is best treated in the linear form. Some of the properties of the linear equation are, however, more readily derived from the non-linear form (5), which might be called the *cognate equation* of the first order. For example, if from the equation

$$\frac{dy}{dx} + y^2 + Py + Q = 0, \tag{5}$$

we wish to remove the term Py, by a change of dependent variable, we obviously must assume

$$y = v - \tfrac{1}{2}P; \tag{11}$$

and making the substitution we have

$$\frac{dv}{dx} - \frac{1}{2}\frac{dP}{dx} + v^2 - Pv + \tfrac{1}{4}P^2 + Pv - \tfrac{1}{2}P^2 + Q = 0,$$

or

$$\frac{dv}{dx} + v^2 + Q_0 = 0, \tag{12}$$

where

$$Q_0 = Q - \tfrac{1}{4}P^2 - \tfrac{1}{2}\frac{dP}{dx}. \tag{13}$$

Consider now the equation to which (12) is cognate; viz:—

$$\frac{d^2u}{dx^2} + Q_0u = 0, \tag{14}$$

the relation between u and v being, as in (6),

$$v = \frac{d}{dx}\log u.$$

Equation (11) gives $\log w = \log u - \tfrac{1}{2}\int Pdx,$

so that $$w = e^{-\frac{1}{2}\int P dx}u,$$

is the transformation which converts

$$\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Q = 0 \qquad (1)$$

into the form (14) which lacks the second term, and Q_0 has the value given in equation (13).

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SOLUTIONS OF EXERCISES.

60 and 61

DIVIDE a sphere by a secant plane into two parts whose surfaces (volumes) shall be as 1 : 3.

SOLUTION.

Let the secant plane divide the diameter normal to it in the ratio $m : n$. The heights and surfaces of the two zones are respectively

$$\frac{n}{m+n}D, \quad \frac{m}{m+n}D; \quad \text{and} \quad \frac{\pi n}{m+n}D^2, \quad \frac{\pi m}{m+n}D^2.$$

The area of the common base of the two segments is

$$\frac{\pi mn}{(m+n)^2}D^2.$$

Which values under the given condition, give

1. For the surfaces,

$$3\left[\frac{\pi nD^2}{m+n} + \frac{\pi mnD^2}{(m+n)^2}\right] = \frac{\pi mD^2}{m+n} + \frac{\pi mnD^2}{(m+n)^2};$$

from which $$\left(\frac{m}{n}\right)^2 - 4\left(\frac{m}{n}\right) - 3 = 0,$$

$$\frac{m}{n} = \sqrt{7} + 2 = 4.64575 + .$$